BASES FOR TOPOLOGIES

DEFINITION: A **base** $\mathcal{B}$ for a topology $\mathcal{T}$ on a set X is a collection $\mathcal{B} \subseteq \mathcal{T}$ such that:

$$\forall T \in \mathcal{T},\, x \in T \implies \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq T$$

Said differently, $\forall T \in \mathcal{T}$, T is a union of elements of $\mathcal{B}$.

NOTE: The elements of $\mathcal{B}$ are called **basic open sets**.

EXAMPLES:

- Every topology $\mathcal{T}$ is a base for itself.
- For a nonempty set X, the collection $\mathcal{B} = \{\{x\} : x \in X\}$ is a base for the discrete topology on X.
- The collection $\{(a, b) : a, b \in \mathbb{R},\, a < b\}$ is a base for the Euclidean topology on $\mathbb{R}$.
- The collection $\{(a, b) : a, b \in \mathbb{Q},\, a < b\}$ is a base for the Euclidean topology on $\mathbb{R}$.
- The collection $\{[a, b) : a, b \in \mathbb{R},\, a < b\}$ is a base for the Sorgenfrey topology on $\mathbb{R}$.
- Is the collection $\{[a, b) : a, b \in \mathbb{Q},\, a < b\}$ a base for the Sorgenfrey topology on $\mathbb{R}$?

THEOREM: Let X be a nonempty set and let $\mathcal{B} \subseteq \mathcal{P}(X)$.

Let $\mathcal{T}$ to be the collection of all sets formed by taking (arbitrary) unions of elements of $\mathcal{B}$.

Then $\mathcal{T}$ is a topology iff

- $\forall x \in X$, $\exists B \in \mathcal{B}$ such that $x \in B$. (i.e., $\mathcal{B}$ **covers** X)
- If $B_1, B_2 \in \mathcal{B}$, $x \in B_1 \cap B_2 \implies \exists B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

In this case, $\mathcal{T}$ is called the topology **generated by** $\mathcal{B}$.

NOTE: $\mathcal{T}$ **weakest** topology which contains $\mathcal{B}$. (Can you prove this?)

THEOREM: Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are topological spaces and $\mathcal{B}$ is a base for $\mathcal{U}$.

A function $F : X \to Y$ is continuous iff $F^{-1}(B) \in \mathcal{T}$ for all $B \in \mathcal{B}$.

In other words, F is continuous iff pre-images of **basic** open sets are open.